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A note on powers of ideals

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Abstract

In this paper we prove that if R is a Noetherian local ring and I is an ideal of R, then I^n can be generated by a quadratic sequence for all sufficiently large n. As an application, we show that if R is a Noetherian ring and I is an ideal of R, then I^n is of quadratic type for all sufficiently large n. © 1998 Elsevier Science B.V.

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1. Introduction

Let R be a Noetherian ring, an ordered sequence $x_1,\ldots,x_n$ of a ring R is said to be a d-sequence if either (both) of the following equivalent conditions hold:

(1) $((x_1,\ldots,x_{i-1}):x_ix_k) = ((x_1,\ldots,x_{i-1}):x_k)$ for all $1 \le i \le n$ and for all $k \ge i$,

(2) $((x_1,\ldots,x_{i-1}):x_i) \cap (x_1,\ldots,x_n) = (x_1,\ldots,x_{i-1})$ for all $1 \le i \le n$,

where $(x_1,\ldots,x_{i-1})$ is interpreted as 0 when $i = 1$. This concept was introduced by Huneke [2] in order to calculate the asymptotic value of depth(R/I^n) for some specific ideal I of R. He later developed the theory of weak d-sequences [3] and used it effectively to calculate the asymptotic value of depth(R/I^n) for an even larger class of examples of ideals I. As the terminology suggests, d-sequences are weak d-sequences. Recently, Raghavan's work about quadratic sequences [6] simplifies and extends the theory of d-sequences and weak d-sequences. These sequences have been shown to have many nice properties; for instance, ideals generated by d-sequences are of linear type [2]; ideals generated by quadratic sequences are of quadratic type [7]. In his thesis [5], Martin explored a connection between d-sequences and quadratic sequences,

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namely, he proved that if I is an ideal generated by a d-sequence, then all powers of I are generated by quadratic sequences This motivates us to ask the following question.

Question. Let R be a Noetherian ring, and I be an ideal of R. Then does there exist an integer N such that for all $n \geq N$ I^n can be generated by a quadratic sequence?

Our main result Theorem 3.8 and Corollary 3.10 provide two classes of rings for which the answer of the question is positive, namely, if R is a Noetherian local ring with infinite residue field or if $R = S[X,X^{-1}]$, where S is a Noetherian ring and X is an indeterminate.

As applications, we shall reprove in Section 4 the following results.

Theorem 4.1 (Johnston and Katz [4]). *Let R be a Noetherian ring and I be an ideal of R; then $\forall n \gg 0$, $reltype(I^n) \leq 2$.*

Theorem 4.2 (Brodmann [1]). *Let R be a Noetherian ring and I be an ideal of R; then for n sufficiently large, $Ass(R/I^n)$ is independent of n. In particular, $\bigcup_{n\geq 1} Ass(R/I^n)$ is finite.*

The author thanks Craig Huneke for pointing out the result of Martin's.

2. Some definitions

In this section, we shall recall some definitions.

Definition 2.1. Let R be a Noetherian ring, $I = (x_1,\dots,x_n)$ be an ideal of R. Let

$$R[It] = \bigoplus_{n\geq 0} I^n t^n$$

be the Rees algebra associated to I; then there is a canonical surjection $\phi : R[X_1,\dots,X_n] \longrightarrow R[It]$ given by $X_i \longmapsto x_i t$. The relation type of I, denoted by $reltype(I)$ is defined to be the least integer r so that $Ker\ \phi$ has a generating set whose elements have degrees at most r.

Remark. It is well-known that the invariant $reltype(I)$ only depends on the ideal I and is independent of the choice of a generating set of I. (cf. [6])

If $reltype(I) \leq 1$, then we say that I is of linear type. If $reltype(I) \leq 2$, then I is said to be of quadratic type.

Before giving the definition of quadratic sequence, we adopt some notations from [6].

By a poset, we mean a partially ordered set. Let Λ be a finite poset. A subset Σ of Λ is said to be an ideal if it satisfies the following property:

$$\text{if } \sigma \in \Sigma,\ \lambda \in \Lambda, \text{ and } \lambda \leq \sigma, \text{ then } \lambda \in \Sigma.$$

Let R be a ring and $\{x_\lambda \mid \lambda \in \Lambda\}$ be a set of elements of R indexed by Λ. Given an ideal Σ of Λ, we denote by X_Σ, the ideal generated by $\{x_\sigma \mid \sigma \in \Sigma\}$, and call it the Λ-ideal (of R) defined by Σ. The empty subset of Λ is an ideal and the Λ-ideal defined by it is the ideal 0. Let $X = X_\Lambda$, the ideal generated by $\{x_\lambda \mid \lambda \in \Lambda\}$.

Given an ideal Σ and an element λ of Λ, we say that λ lies just above Σ if $\lambda \notin \Sigma$ but every element $\sigma \in \Sigma$ such that $\sigma < \lambda$ belongs to Σ. We say that λ lies inside or just above Σ if it is either belongs to Σ or is just above Σ.

Definition 2.2 (*cf. Raghavan* [6]). We say that $\{x_\lambda \mid \lambda \in \Lambda\}$ is a quadratic sequence if, for every pair (Σ, λ) where Σ is an ideal of Λ and λ is an element of Λ that lies just above (equivalently, lies inside or just above) Σ, there exists an ideal Θ of Λ such that

(1) $(X_\Sigma : x_\lambda) \cap X \subseteq X_\Theta$ and
(2) $x_\lambda X_\Theta \subseteq X_\Sigma X$.

One of the important result in [7] is that ideals generated by quadratic sequences are of quadratic type.

Our primary tool in this article will be the notion of *superficial elements.*

Definition 2.3. Let R be a Noetherian ring, I be an ideal of R, and $G(I) = \bigoplus_{n \geq 0} I^n/I^{n+1}$ be the associated graded ring of I. An element $x \in R$ is called a superficial element for I of degree k if $x \in I^k \setminus I^{k+1}$ such that $(0 :_{G(I)} x^*) \cap G_n = 0$ for all n sufficiently large, where $x^* = x + I^{k+1}$ is the leading form of x in $G(I)$ and $G_n = I^n/I^{n+1}$ is the n-th component of $G(I)$.

We would like to mention the following important result in [9, p. 287].

Theorem 2.4. *If $(R, \mathfrak{m})$ is a Noetherian local ring with an infinite residue field and I is an $\mathfrak{m}$-primary ideal of R, then there exists a superficial element for I of degree* 1.

3. Powers of ideals and quadratic sequences

We begin this section by proving the following theorem.

Theorem 3.1. *Let R be a ring and $I = (x_1, \ldots, x_t)$. Let $R_0 = R$ and $R_i = R/(x_1, \ldots, x_i)$ for $i = 1, \ldots, t-1$. Suppose that there exists an $N > 0$ such that*

(i) $\forall 1 \leq i \leq t$,

$$((0) : x_i^n R_{i-1}) \cap I^n R_{i-1} = 0 \quad \forall n \geq N.$$

(ii) $\forall 1 \leq i \leq t$,

$$I^n R_{i-1} : x_i R_{i-1} = I^{n-1} R_{i-1} + (0) : x_i R_{i-1} \quad \forall n \geq N.$$

Then for every $n \geq N$, $\{x_1^n, x_1^{n-1}x_2, \ldots, x_1^{n-1}x_t, \ldots, x_t^n\}$ *form a quadratic sequence under the reverse lexicographic order*

$$x_1^n < x_1^{n-1}x_2 < \cdots < x_1^{n-1}x_t < \cdots < x_t^n.$$

Proof. Let $n \geq N$ be fixed. Let $T = \{x_1^n, x_1^{n-1}x_2, \ldots, x_1^{n-1}x_t, \ldots, x_t^n\}$ be indexed by a poset Λ (In fact, a totally ordered set) so that $T = \{x_\lambda \mid \lambda \in \Lambda\}$ and $\sigma < \lambda$ if and only if x_σ is smaller than x_λ under the reverse lexicographic order.

Let $n_1, \ldots, n_t$ be nonnegative integers such that $n_1 + \cdots + n_t = n$; then there is an index $\lambda \in \Lambda$ such that $x_\lambda = x_1^{n_1} \cdots x_t^{n_t}$. Let $\Sigma = \{\sigma \in \Lambda \mid \sigma < \lambda\}$; then the Λ-ideal X_Σ is generated by elements in T which are smaller than $x_1^{n_1} \cdots x_t^{n_t}$ under the reverse lexicographic order. To show Theorem 3.1, it is sufficient to show that there is an ideal Θ of Λ such that

(1) $(X_\Sigma : x_\lambda) \cap X \subseteq X_\Theta$, and

(2) $x_\lambda X_\Theta \subseteq X_\Sigma X$.

Let $1 \leq s \leq t$ be the largest integer so that $n_s \neq 0$. Let $J = (x_1, \ldots, x_{s-1})I^{n-1}$ if $s \geq 2$ and $J = 0$ if $s = 1$; then J is generated by the elements in T which are smaller than x_s^n under the reverse lexicographic order, so that $J = X_\Theta$ for some ideal Θ of Λ and J is an Λ ideal. We shall show in the following that J satisfies the statements (1) and (2).

If $s = 1$, then $n_1 = n$, $X_\Sigma = 0$ and $J = 0$. Since $((0) : x_1^n) \cap I^n = 0$, (1′) and (2′) hold.

Assume $s \geq 2$. Notice that

$$\begin{aligned} x_1^{n_1} \cdots x_s^{n_s}(x_1, \ldots, x_{s-1})I^{n-1} &= x_1^{n_1+1}x_2^{n_2} \cdots x_{s-1}^{n_{s-1}}x_s^{n_s-1}(x_sI^{n-1}) \\ &\quad + x_1^{n_1}x_2^{n_2+1}x_3^{n_3} \cdots x_{s-1}^{n_{s-1}}x_s^{n_s-1}(x_sI^{n-1}) \\ &\quad + \cdots + x_1^{n_1} \cdots x_{s-2}^{n_{s-2}}x_{s-1}^{n_{s-1}+1}x_s^{n_s-1}(x_sI^{n-1}) \end{aligned}$$

and

$$\{x_1^{n_1+1}x_2^{n_2} \cdots x_{s-1}^{n_{s-1}}x_s^{n_s-1}, \ldots, x_1^{n_1} \cdots x_{s-2}^{n_{s-2}}x_{s-1}^{n_{s-1}+1}x_s^{n_s-1}\} \subseteq \{x_\sigma \mid \sigma \in \Sigma\},$$

we see that $Jx_\lambda \subseteq X_\Sigma X$ and (2) is satisfied. Moreover, (2) implies that

$$(X_\Sigma : x_\lambda) \cap X = J + (X_\Sigma : x_\lambda) \cap (x_s, \ldots, x_t)^n$$

as $X = J + (x_s, \ldots, x_t)^n$. Therefore, to show (1), it remains to show that

$$(X_\Sigma : x_\lambda) \cap (x_s, \ldots, x_t)^n \subseteq J.$$

Let $r \in (x_s, \ldots, x_t)^n$ such that $rx_1^{n_1} \cdots x_s^{n_s} \in X_\Sigma$. Observe that an element $x_1^{l_1} \cdots x_t^{l_t} \in T$ belongs to $\{x_\sigma \mid \sigma \in \Sigma\}$ if and only if there is an $i < s$ such that $l_j = n_j$ for $j < i$ and $l_i \geq n_i + 1$. Therefore,

$$X_\Sigma \subseteq (x_1^{n_1+1}) + (x_1^{n_1}x_2^{n_2+1}) + \cdots + (x_1^{n_1} \cdots x_{s-2}^{n_{s-2}}x_{s-1}^{n_{s-1}+1}).$$

In particular, there are $r_i \in R, i = 1,\dots,s-1$ such that

$$rx_1^{n_1}\cdots x_s^{n_s} = r_1x_1^{n_1+1} + r_2x_1^{n_1}x_2^{n_2+1} + \cdots + r_{s-1}x_1^{n_1}\cdots x_{s-2}^{n_{s-2}}x_{s-1}^{n_{s-1}+1}.$$

Multiplying by x_s^n, we obtain

$$rx_1^{n_1}\cdots x_{s-1}^{n_{s-1}}x_s^{n_s+n} = r_1x_1^{n_1+1}x_s^n + r_2x_1^{n_1}x_2^{n_2+1}x_s^n + \cdots + r_{s-1}x_1^{n_1}\cdots x_{s-2}^{n_{s-2}}x_{s-1}^{n_{s-1}+1}x_s^n.$$

Claim. Let $0 \le i \le s-1$; then in R_i,

$$rx_{i+1}^{n_{i+1}}\cdots x_{s-1}^{n_{s-1}}x_s^{n_s+n} = r_{i+1}x_{i+1}^{n_{i+1}+1}x_s^n + r_{i+2}x_{i+1}^{n_{i+1}}x_{i+2}^{n_{i+2}+1}x_s^n + \cdots + r_{s-1}x_{i+1}^{n_{i+1}}\cdots x_{s-2}^{n_{s-2}}x_{s-1}^{n_{s-1}+1}x_s^n.$$

We prove the claim by induction on i.

If $i = 0$, then there is nothing to prove.

Assume that the claim holds for some i, $0 \le i \le s-2$; then in R_i

$$x_{i+1}^{n_{i+1}}L = 0,$$

where

$$L = rx_{i+2}^{n_{i+2}}\cdots x_{s-1}^{n_{s-1}}x_s^{n_s+n} - r_{i+1}x_{i+1}x_s^n - r_{i+2}x_{i+2}^{n_{i+2}+1}x_s^n - \cdots - r_{s-1}x_{i+2}^{n_{i+2}}\cdots x_{s-2}^{n_{s-2}}x_{s-1}^{n_{s-1}+1}x_s^n.$$

It follows that $L \in ((0) : x_{i+1}^nR_i) \cap I^nR_i = 0$ by assumption, and therefore in R_{i+1}

$$rx_{i+2}^{n_{i+2}}\cdots x_{s-1}^{n_{s-1}}x_s^{n_s+n} = r_{i+2}x_{i+2}^{n_{i+2}+1}x_s^n + \cdots + r_{s-1}x_{i+2}^{n_{i+2}}\cdots x_{s-2}^{n_{s-2}}x_{s-1}^{n_{s-1}+1}x_s^n.$$

This completes the proof of the claim.

By setting $i = s-1$ in the claim, we obtain that $rx_s^{n_s+n} = 0$ in R_{s-1}. Since $r \in I^n$ and $((0) : x_s^{2n}R_{s-1}) \cap I^nR_{s-1} = 0$, $r = 0$ in R_{s-1}. So we have

$$r \in (x_s,\dots,x_t)^n \cap (x_1,\dots,x_{s-1}) \subseteq I^n \cap (x_1,\dots,x_{s-1}).$$

Thus, $r \in J$ by the following lemma. This completes the proof. □

Lemma 3.2. *Let R be a Noetherian ring, $t \ge 2$ and $I = (x_1,\dots,x_t)R$. Let $R_0 = R$ and $R_i = R/(x_1,\dots,x_i)$ for $i = 1,\dots,t-1$. Assume that for some $n > 0$ and $\forall 1 \le i \le t-1$*

$$I^nR_{i-1} : x_iR_{i-1} = I^{n-1}R_{i-1} + (0) : x_iR_{i-1}.$$

Then $\forall j = 1,\dots,t-1$

$$I^n \cap (x_1,\dots,x_j) = (x_1,\dots,x_j)I^{n-1}.$$

Proof. We proceed by induction on j.

If $j = 1$, then

$$I^n \cap (x_1) = x_1(I^n : x_1) \subseteq x_1(I^{n-1} + (0) : x_1) = x_1 I^{n-1}.$$

Assume $j \geq 2$. Notice that by assumption,

$$\begin{aligned} I^n R_{j-1} \cap x_j R_{j-1} &\subseteq x_j(I^n R_{j-1} : x_j R_{j-1}) \\ &= x_j(I^{n-1} R_{j-1} + (0) : x_j R_{j-1}) \\ &= x_j I^{n-1} R_{j-1}. \end{aligned}$$

Therefore, we have

$$I^n \cap (x_1, \ldots, x_j) \subseteq x_j I^{n-1} + (x_1, \ldots, x_{j-1}).$$

However by induction, we have

$$I^n \cap (x_1, \ldots, x_{j-1}) \subseteq (x_1, \ldots, x_{j-1}) I^{n-1},$$

it follows that

$$\begin{aligned} I^n \cap (x_1, \ldots, x_j) &\subseteq x_j I^{n-1} + I^n \cap (x_1, \ldots, x_{j-1}) \\ &= x_j I^{n-1} + (x_1, \ldots, x_{j-1}) I^{n-1} \\ &= (x_1, \ldots, x_j) I^{n-1}. \qquad \square \end{aligned}$$

Observe that a necessary condition for an ideal I satisfying (ii) in Theorem 3.1 is that I has a superficial element of degree 1. Therefore, rings having the property that every ideal has a superficial element of degree 1 will be our first consideration. We shall prove as follows that two classes of rings have such property.

By modifying the original proof of Theorem 2.4, it is easy to obtain the following.

Lemma 3.3. *Let $(R, \mathfrak{m})$ be a Noetherian local ring with an infinite residue field, and I be an ideal of R. Then for any finite family $\{I_\alpha\}$ of ideals of R such that $I \not\subseteq I_\alpha$ $\forall \alpha$, there is an element $x \in I \setminus \mathfrak{m} I$ such that x is a superficial element for I and $x \notin I_\alpha \forall \alpha$.*

Lemma 3.4. *Let R be a Noetherian ring containing an infinite subset Λ with the following properties*:

(1) *Every element in Λ is an unit.*

(2) $\forall n \geq 1$, *the subset $S(\Lambda, n)$ of R consists of only units and* 0, *where $S(\Lambda, n)$ is the set of the determinants of all the $n \times n$ matrices with entries in Λ.*

Let I be an ideal of R generated by $\{x_1, \ldots, x_t\}$; then for any finite family $\{I_\alpha\}$ of ideals of R such that $I \not\subseteq I_\alpha$ $\forall \alpha$, there is an element $x \in I$ of the form $\sum_{i=1}^t \lambda_i x_i$ with $\lambda_i \in \Lambda$ such that x is a superficial element for I and $x \notin I_\alpha \forall \alpha$.

An immediate corollary of the above lemmas is:

Corollary 3.5. *Let R be as in Lemma 3.3 or in Lemma 3.4 and I be an ideal of R. Then there is an element $x \in I \setminus \mathfrak{m}I$ such that*

$$\text{(i)} \qquad ((0):x^n) \cap I^n = (0) \quad \forall n \gg 0.$$

and

$$\text{(ii)} \qquad I^n : x = I^{n-1} + ((0):x) \quad \forall n \gg 0$$

Proof. Let R be as in Lemma 3.3 or in Lemma 3.4 and I be an ideal of R. Let I_α be the set of the associated prime ideals that do not contain I; then it is easy to verify that if $y \in I \setminus \cup I_\alpha$, then $((0):y^n) \cap I^n = (0)$ $\forall n \gg 0$. Hence by Lemma 3.3 or Lemma 3.4, I has a superficial element x of degree 1 such that (i) holds. Moreover, since x is a superficial element for I, there is a positive integer c such that $(I^n : x) \cap I^c = I^{n-1}$ for all $n > c$. On the other hand, by Artin-Rees Lemma, there is an integer $k > 0$ such that $I^n \cap (x) \subseteq xI^{n-k}$ $\forall n \geq k$. Therefore, for $n > k + c$ and $a \in (I^n : x)$, we have $ax \in I^n \cap (x) \subseteq xI^{n-k} \subseteq xI^c$, it follows that $a \in I^c + (0):x$. Hence for $n > k + c$

$$(I^n : x) = (I^n : x) \cap (I^c + (0):x) = (I^c \cap (I^n : x)) + (0):x = I^{n-1} + (0):x. \qquad \square$$

Notice that if R is a local ring with infinite residue field and $x \in I \setminus \mathfrak{m}I$ is a superficial element for I of degree 1, then $\bar{R} = R/(x)$ is a Noetherian local ring with an infinite residue field and $\mu(\bar{I}) = \mu(I) - 1$, so that by induction on $\mu(I)$ and by applying Corollary 3.5, we shall obtain the following.

Corollary 3.6. *Suppose R is a Noetherian local ring with an infinite residue field. Then for any ideal I of R, there are $x_1, \ldots, x_t \in I$ such that $I = (x_1, \ldots, x_t)$ and such that*

(i) $\forall 1 \leq i \leq t$,

$$((0):x_i^n R_{i-1}) \cap I^n R_{i-1} = 0 \quad \forall n \gg 0$$

and

(ii) $\forall 1 \leq i \leq t$,

$$I^n R_{i-1} : x_i R_{i-1} = I^{n-1} R_{i-1} + (0) : x_i R_{i-1} \quad \forall n \gg 0,$$

where $R_0 = R$ and $R_i = R/(x_1, \ldots, x_i)$ for $i = 1, \ldots, t-1$.

For rings satisfying the assumptions in Lemma 3.4, we also have the same conclusion:

Corollary 3.7. *Let R be as in Lemma 3.4. Then for any ideal I of R, there are $x_1, \ldots, x_t \in I$ such that $I = (x_1, \ldots, x_t)$ and that*

(i) $\forall 1 \leq i \leq t$,

$$((0):x_i^n R_{i-1}) \cap I^n R_{i-1} = 0 \quad \forall n \gg 0$$

and

(ii) $\forall 1 \le i \le t$,

$$I^n R_{i-1} : x_i R_{i-1} = I^{n-1} R_{i-1} + (0) : x_i R_{i-1} \quad \forall n \gg 0,$$

where $R_0 = R$ *and* $R_i = R/(x_1, \dots, x_i)$ *for* $i = 1, \dots, t-1$.

Proof. Let $\{x_1, \dots, x_t\}$ be a generating set of I; then by Corollary 3.5 there is an element $x \in I$ of the form $\sum_{i=1}^{t} \lambda_i x_i$ with $\lambda_i \in \Lambda$ satisfies (i) and (ii) of Corollary 3.5. Let $\bar{R} = R/(x)$; then $\bar{I} = (\bar{x}_2, \dots, \bar{x}_t)$, so that by induction on t, we shall obtain the assertion provided that the image $\bar{\Lambda}$ of Λ in $\bar{R}$ is infinite and satisfies (1) and (2). However, since I is proper, (1) and (2) hold trivially. Also, from the fact that $S(\Lambda, 2)$ consists of units and 0, we see that if $\lambda_1 \ne \lambda_2$, then $\lambda_1 - \lambda_2$ is an unit, so that $\bar{\lambda}_1 \ne \bar{\lambda}_2$. It follows that $\bar{\Lambda}$ is infinite. □

What Corollaries 3.6, 3.7 and Theorem 3.1 imply are our main results.

Theorem 3.8. *Suppose R is a Noetherian local ring with an infinite residue field. Let I be an ideal of R; then* $\forall n \gg 0$, I^n *can be generated by a quadratic sequence.*

Theorem 3.9. *Let R be a Noetherian ring containing an infinite subset* Λ *with the following properties*:

(1) *Every element in* Λ *is an unit.*

(2) $\forall n \ge 1$, *the subset* $S(\Lambda, n)$ *of R contains only units and* 0, *where* $S(\Lambda, n)$ *is the set of the determinants of all the* $n \times n$ *matrices with entries in* Λ.

Let I be an ideal of R; then $\forall n \gg 0$, I^n *can be generated by a quadratic sequence.*

Let R be a Noetherian ring and X be an indeterminant; then $\Lambda = \{X^{-n} \mid n > 0\}$ is an infinite subset of $R[X, X^{-1}]$ satisfies (1) and (2) in Theorem 3.9, so that we have

Corollary 3.10. *Let R be a Noetherian ring and X be an indeterminant; then for every ideal I of* $R[X, X^{-1}]$ *and for all n sufficiently large,* I^n *can be generated by a quadratic sequence* $\forall n \gg 0$.

The following corollary is an extension of [5, Theorem 5.2.3].

Corollary 3.11. *Let R be a ring and let* $\{x_1, \dots, x_t\}$ *be a quadratic sequence under the natural order:* $x_1 < x_2 < \cdots < x_t$. *Let* $I = (x_1, \dots, x_t)R$; *then* $\forall n \ge 1$ $\{x_1^n, x_1^{n-1} x_2, \dots, x_1^{n-1} x_t, \dots, x_t^n\}$ *form a quadratic sequence under the reverse lexicographic order*

$$x_1^n < x_1^{n-1} x_2 < \cdots < x_1^{n-1} x_t < \cdots < x_t^n.$$

Proof. Since $\{x_1, \dots, x_t\}$ is a quadratic sequence by assumption, the assertion holds for $n = 1$, so we may assume that $n \ge 2$.

By Theorem 3.1, we need only to show that the integer N in Theorem 3.1 can be chosen to be 2. That is, the following hold:

(i) $\forall 1 \le i \le t$,

$$((0) : x_i^n R_{i-1}) \cap I^n R_{i-1} = 0 \quad \forall n \ge 2.$$

(ii) $\forall 1 \le i \le t$,

$$I^n R_{i-1} : x_i R_{i-1} = I^{n-1} R_{i-1} + (0) : x_i R_{i-1} \quad \forall n \ge 2,$$

where $R_0 = R$ and $R_i = R/(x_1, \ldots, x_i)$ for $i = 1, \ldots, t-1$.

Notice that if $\{x_\lambda \mid \lambda \in \Lambda\}$ is a quadratic sequence and x is the smallest element of $\{x_\lambda \mid \lambda \in \Lambda\}$, then by Definition 2.2 $((0):x) \cap (x) \subseteq ((0):x) \cap X = 0$, hence $(0):x^n = (0):x$, it follows that

$$((0):x^n) \cap X^n = ((0):x) \cap X^n \subseteq ((0):x) \cap X = 0,$$

$\forall n \ge 2$. Furthermore, by [6, Corollary 9.7], we have $(x) \cap X^n = xX^{n-1} \quad \forall n \ge 1$.

Since $\forall i = 1, \ldots, t$, the images of $\{x_i, \ldots, x_t\}$ in R_{i-1} also forms a quadratic sequence by [6, Remark 9.4], by applying the above criterion on $\{x_i, \ldots, x_t\}$ in R_{i-1}, we can see easily that (i) and (ii) hold. □

4. Applications

By using the results we obtained in Section 3, we are able to show the following.

Theorem 4.1 (Johnston and Katz [4]). *Let R be a Noetherian ring and I be an ideal of R; then $\forall n \gg 0$, reltype$(I^n) \le 2$.*

Proof. Since $R \longrightarrow R[X, X^{-1}]$ is a faithful flat map, by [8, Lemma 4.1], *reltype*(I^n) = *reltype*$(I^n S)$, where $S = R[X, X^{-1}]$. Therefore, we have to show that *reltype*$(I^n S) \le 2$ for $n \gg 0$. However, this simply follows from Corollary 3.10 and [6, Corollary 9.8]. □

Theorem 4.2 (Brodmann [1]). *Let R be a Noetherian ring and I be an ideal of R. Then Ass(R/I^n) is independent of n for sufficiently large n. In particular, $\bigcup_{n \ge 1}$Ass (R/I^n) is finite.*

We need the following lemma.

Lemma 4.3. *Let R be a ring and I be an ideal of R. Suppose that there is an element $x \in I$ such that*

(i) $\quad ((0):x^n) \cap I^n = 0 \quad \forall n \ge N$

and

(ii) $\quad I^n : x = I^{n-1} + (0) : x \quad \forall n \geq N$

for some integer $N > 0$. *Then*

$$Ass(R/I^N) \subseteq Ass(R/I^{N+1}) \subseteq Ass(R/I^{N+2}) \subseteq \cdots.$$

Proof. Let $n \geq N$; then by (ii)

$$0 \longrightarrow R/(I^n + (0) : x) \xrightarrow{x} R/I^{n+1}$$

is exact, so that $Ass(R/(I^n + (0) : x)) \subseteq Ass(R/I^{n+1})$. Furthermore, since

$$0 \longrightarrow \frac{(I^n + (0) : x)}{I^n} \longrightarrow R/I^n \longrightarrow R/(I^n + (0) : x) \longrightarrow 0$$

is exact, and by (i)

$$\frac{(I^n + (0) : x)}{I^n} \cong \frac{(0) : x}{((0) : x) \cap I^n} = (0) : x,$$

we obtain that

$$\begin{aligned} Ass(R/I^n) &\subseteq Ass\left(\frac{(I^n + (0) : x)}{I^n}\right) \cup Ass(R/(I^n + (0) : x)) \\ &\subseteq Ass((0) : x) \cup Ass(R/I^{n+1}). \end{aligned}$$

However, we also have

$$\frac{(I^{n+1} + (0) : x)}{I^{n+1}} \cong (0) : x,$$

it follows that

$$Ass((0) : x) = Ass\left(\frac{(I^{n+1} + (0) : x)}{I^{n+1}}\right) \subseteq Ass(R/I^{n+1}),$$

and $Ass(R/I^n) \subseteq Ass(R/I^{n+1})$. $\square$

Proof of Theorem 4.2. Let $S = R[X, X^{-1}]$. Notice that if J is an ideal of R and $\{P_j\}$ is the set of the associative prime ideals of J, then $\{P_jS\}$ is the set of the associative prime ideals of JS. Therefore, we need only to show the assertion for S.

Let I be an ideal of S; then by Corollary 3.7 there is a generating set $\{x_1, \ldots, x_t\}$ of I and an integer $N > 0$ such that

(i) $\forall 1 \leq i \leq t$,

$$((0) : x_i^n R_{i-1}) \cap I^n R_{i-1} = 0 \quad \forall n \geq N$$

and

(ii) $\forall 1 \le i \le t$,

$$I^n R_{i-1} : x_i R_{i-1} = I^{n-1} R_{i-1} + (0) : x_i R_{i-1} \quad \forall n \ge N,$$

where $R_0 = R$ and $R_i = R/(x_1, \ldots, x_i)$ for $i = 1, \ldots, t-1$.

Claim. $\forall n \ge N$

$$Ass(R/I^{n+2}) \setminus Ass(R/I^{n+1}) \subseteq Ass(R_1/I^{n+2}R_1).$$

Let $n \ge N$; then by (i)

$$\frac{I^n + (0) : x_1}{I^{n+1} + (0) : x_1} \cong \frac{I^n}{I^n \cap (I^{n+1} + (0) : x_1)} = \frac{I^n}{I^{n+1} + I^n \cap ((0) : x_1)} = \frac{I^n}{I^{n+1}}.$$

Furthermore, by (ii)

$$0 \longrightarrow R/(I^n + (0) : x_1) \xrightarrow{x_1} R/(I^{n+1})$$

is exact, so that from the short exact sequence

$$0 \longrightarrow \frac{I^n + (0) : x_1}{I^{n+1} + (0) : x_1} \longrightarrow R/(I^{n+1} + (0) : x_1) \longrightarrow R/(I^n + (0) : x_1) \longrightarrow 0,$$

we obtain

$$Ass(R/(I^{n+1} + (0) : x_1)) \subseteq Ass\left(\frac{I^n}{I^{n+1}}\right) \cup Ass(R/(I^n + (0) : x_1)) \subseteq Ass(R/I^{n+1}).$$

Moreover, since

$$0 \longrightarrow R/(I^{n+1} + (0) : x_1) \xrightarrow{x_1} R/(I^{n+2}) \longrightarrow R_1/I^{n+2}R_1 \longrightarrow 0$$

is exact, we see that

$$\begin{aligned} Ass(R/I^{n+2}) \setminus Ass(R/I^{n+1}) &\subseteq Ass(R/I^{n+2}) \setminus Ass(R/(I^{n+1} + (0) : x_1)) \\ &\subseteq Ass(R_1/I^{n+2}R_1), \end{aligned}$$

and the proof of the claim is complete.

We now prove the assertion by induction on t. If $t = 1$, then by the claim, $Ass(R/I^{n+2}) \setminus Ass(R/I^{n+1}) \subseteq Ass(R/(x_1))$, $\forall n \ge N$. Since $Ass(R/(x_1))$ is finite and, by Lemma 4.3,

$$Ass(R/I^N) \subseteq Ass(R/I^{N+1}) \subseteq Ass(R/I^{N+2}) \subseteq \cdots,$$

the assertion holds.

If $t \ge 2$, then again by the claim we have $Ass(R/I^{n+2}) \setminus Ass(R/I^{n+1}) \subseteq Ass(R_1/I^{n+2}R_1) \subseteq \bigcup_{n \ge 1} Ass(R_1/I^n R_1)$, $\forall n \ge N$. Since, by induction, $\bigcup_{n \ge 1} Ass(R_1/I^n R_1)$ is finite, the assertion follows by Lemma 4.3. $\square$

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